

Hamiltonian perspective on generalized complex structure

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ABSTRACT

In this note we clarify the relation between extended world-sheet supersymmetry and generalized complex structure. The analysis is based on the phase space description of a wide class of sigma models. We point out the natural isomorphism between the group of orthogonal automorphisms of the Courant bracket and the group of local canonical transformations of the cotangent bundle of the loop space. Indeed this fact explains the natural relation between the world-sheet and the geometry of $T \oplus T^*$. We discuss D-branes in this perspective.

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1 Introduction

The concept of generalized complex structure was introduced by Hitchin [5] and studied by Gualtieri in his thesis [4]. The generalized complex structure and related constructions such as generalized Kähler and generalized Calabi-Yau structures appear naturally in the context of geometry of the sum of the tangent and cotangent bundles, $T \oplus T^*$. At the same time there are indications that the geometry of $T \oplus T^*$ plays a profound role within modern string theory. Actually before the Hitchin's work [5] some of the relevant mathematical notions were anticipated in the string literature (e.g., the algebraic definition of a generalized complex (Kähler) geometry is discussed in [7]). This note intends to further examine the relation between the geometry of $T \oplus T^*$ and string theory.

In particular we want to explore the relation between the generalized complex geometry and extended supersymmetry of world-sheet theories. The objective of this note is to clarify and extend some of the results from [12]. Typically the extended supersymmetry for the low dimensional sigma models is related to complex geometry and this is a model independent statement. Let us recall the simple algebraic argument for this fact. We start from the ansatz

$$\delta(\epsilon)\Phi^\mu = \epsilon D\Phi^\nu J^\mu_\nu, \quad (1.1)$$

where Φ is a superfield corresponding to the map $X : \Sigma \rightarrow M$. A simple calculation of the algebra gives the following expression

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\Phi^\mu = -2\epsilon_1\epsilon_2\partial\Phi^\lambda(J^\mu_\nu J^\nu_\lambda) - 2\epsilon_1\epsilon_2 D\Phi^\lambda D\Phi^\rho(J^\nu_{\lambda,\rho}J^\mu_\nu - J^\nu_\rho J^\mu_{\lambda,\nu}). \quad (1.2)$$

To reproduce the supersymmetry algebra

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\Phi^\mu = 2\epsilon_1\epsilon_2\partial\Phi^\mu \quad (1.3)$$

J is thus a complex structure. The main idea of this note is try to repeat this simple algebraic argument in phase space (Φ, S) where S is the momentum conjugated to Φ . In the phase space writing down the ansatz for the transformation is equivalent to the choice of symplectic structure and the generator for the transformation. Using the most general form for the generator for second supersymmetry and the standard (twisted) symplectic structure we arrive to the main result of the paper that the phase space realization of extended supersymmetry is related to generalized complex structure. Unlike [12] all results presented in this note are obtained in a model independent way. Indeed the phase space picture offers a natural explanation of the appearance of the geometry of $T \oplus T^*$ and it agrees with the recent work [1] where the role of the Courant bracket has been discussed in this context.

The note is organized as follows. In section 2 we start by reviewing the standard description of the string phase space in terms of cotangent bundle T^*LM of the loop space LM .

Then we introduce the $N = 1$ version of T^*LM and explain the notation. We point out the natural isomorphism between the group of orthogonal automorphism of the Courant bracket and the group of local canonical transformations of T^*LM (or its supersymmetric version). In section 3 we explain the relation between extended supersymmetry and generalized complex geometry. We also explain how the real Dirac structures may arise in this context. In the following section 4 we deal with D-branes in the present context. We replace the loop space LM by the interval space PM . We define the $N = 1$ version of T^*PM and explore the relation to generalized complex submanifolds. Finally, in section 5 we give a summary of the paper with a discussion of the open problems and the relation of our result to previous results in the literature. There are two Appendices at the end of the paper. In the first Appendix we establish our conventions for $N = 1$ superspace. In the second Appendix the basic facts about $T \oplus T^*$ geometry are stated.

2 Hamiltonian formalism

A wide class of sigma models share the following phase space description (e.g., see [1]). For the world-sheet $\Sigma = S^1 \times \mathbb{R}$ the phase space can be identified with a cotangent bundle T^*LM of the loop space $LM = \{X : S^1 \rightarrow M\}$. Using local coordinates $X^\mu(\sigma)$ and their conjugate momenta $p_\mu(\sigma)$ the standard symplectic form on T^*LM is given by

$$\omega = \int_{S^1} d\sigma \delta X^\mu \wedge \delta p_\mu, \quad (2.4)$$

where δ is de Rham differential on T^*LM . The symplectic form (2.4) can be twisted by a closed three form $H \in \Omega^3(M)$, $dH = 0$ as follows

$$\omega = \int_{S^1} d\sigma (\delta X^\mu \wedge \delta p_\mu + \frac{1}{2} H_{\mu\nu\rho} \partial X^\mu \delta X^\nu \wedge \delta X^\rho), \quad (2.5)$$

where $\partial \equiv \partial_\sigma$ is derivative with respect to σ . For both symplectic structures the following transformation is canonical

$$X^\mu \rightarrow X^\mu, \quad p_\mu \rightarrow p_\mu + b_{\mu\nu} \partial X^\nu \quad (2.6)$$

associated with a closed two form, $b \in \Omega^2(M)$, $db = 0$. There are also canonical transformations which correspond to $Diff(M)$ when X transforms as a coordinate and p as a section of cotangent bundle T^*M . In fact the group of local canonical transformations² for T^*LM

²By local canonical transformation we mean those canonical transformations when new pair (\tilde{X}, \tilde{p}) are given as local expression in terms of the old one (X, p) . For example, in the discussion of T-duality one uses non-local canonical transformations, i.e. \tilde{X} is non-local expression in terms of X .

is a semidirect product of $Diff(M)$ and $\Omega_{closed}^2(M)$. Therefore we come to the following proposition

Proposition 1 *The group of local canonical transformations on T^*LM is isomorphic to the group of orthogonal automorphisms of Courant bracket.*

For the description of orthogonal automorphisms of Courant bracket see Appendix B.

This construction is supersymmetrized in rather straightforward fashion (see Appendix A for superspace conventions). Let $S^{1,1}$ be a "supercircle" with coordinates (σ, θ) . Then the corresponding superloop space is $\mathcal{LM} = \{\Phi : S^{1,1} \rightarrow M\}$. The phase space is given by the cotangent bundle $\Pi T^*\mathcal{LM}$ of \mathcal{LM} , however with reversed parity on the fibers. In local coordinates we have a scalar superfield $\Phi(\sigma, \theta)$ and a conjugate momenta, spinorial superfield $S_\mu(\sigma, \theta)$ with the following expansion

$$\Phi^\mu(\sigma, \theta) = X^\mu(\sigma) + \theta \lambda^\mu(\sigma), \quad S_\mu(\sigma, \theta) = \rho_\mu(\sigma) + \theta p_\mu(\sigma), \quad (2.7)$$

where λ and ρ are fermions (their linear combinations can be related to the standard world-sheet fermions ψ_+ and ψ_-). S is a section of the pullback $X^*(\Pi T^*M)$ of the cotangent bundle of M , considered as an odd bundle. The corresponding symplectic structure on $\Pi T^*\mathcal{LM}$ is

$$\omega = \int_{S^{1,1}} d\sigma d\theta (\delta\Phi^\mu \wedge \delta S_\mu - \frac{1}{2} H_{\mu\nu\rho} D\Phi^\mu \delta\Phi^\nu \wedge \delta\Phi^\rho), \quad (2.8)$$

such that the bosonic part of (2.8) coincides with (2.5). Therefore $C^\infty(\Pi T^*\mathcal{LM})$ carries the structure of super-Poisson algebra. Again as in the purely bosonic case the group of local canonical transformations of $\Pi T^*\mathcal{LM}$ is a semidirect product of $Diff(M)$ and $\Omega_{closed}^2(M)$. The b -transform now is given by

$$\Phi^\mu \rightarrow \Phi^\mu, \quad S_\mu \rightarrow S_\mu - b_{\mu\nu} D\Phi^\nu, \quad (2.9)$$

or in components

$$X^\mu \rightarrow X^\mu, \quad p_\mu \rightarrow p_\mu + b_{\mu\nu} \partial X^\nu + b_{\mu\nu,\rho} \lambda^\nu \lambda^\rho, \quad (2.10)$$

$$\lambda^\mu \rightarrow \lambda^\mu, \quad \rho_\mu \rightarrow \rho_\mu - b_{\mu\nu} \lambda^\nu. \quad (2.11)$$

3 Supersymmetry in phase space

In this section we describe the conditions under which extended supersymmetry can be introduced on $\Pi T^*\mathcal{LM}$. We start from the case $H = 0$. By construction of $\Pi T^*\mathcal{LM}$ the

generator of manifest supersymmetry is given by

$$\mathbf{Q}_1(\epsilon) = - \int_{S^{1,1}} d\sigma d\theta \epsilon S_\mu Q \Phi^\mu, \quad (3.12)$$

where Q is operator introduced in (A.1) and ϵ is an odd parameter. Using (2.8) we can calculate the Poisson brackets for supersymmetry generators

$$\{\mathbf{Q}_1(\epsilon), \mathbf{Q}_1(\tilde{\epsilon})\} = \mathbf{P}(2\epsilon\tilde{\epsilon}), \quad (3.13)$$

where P is generator of translations along σ

$$\mathbf{P}(a) = \int_{S^{1,1}} d\sigma d\theta a S_\mu \partial \Phi^\mu \quad (3.14)$$

with a being an even parameter.

Next we study when there exists a second supersymmetry. The second supersymmetry should be generated by some $\mathbf{Q}_2(\epsilon)$ such that it satisfies the following brackets

$$\{\mathbf{Q}_1(\epsilon), \mathbf{Q}_2(\tilde{\epsilon})\} = 0, \quad \{\mathbf{Q}_2(\epsilon), \mathbf{Q}_2(\tilde{\epsilon})\} = \mathbf{P}(2\epsilon\tilde{\epsilon}). \quad (3.15)$$

By dimensional arguments there is a unique ansatz for the generator $\mathbf{Q}_2(\epsilon)$ on $\Pi T^* \mathcal{LM}$ which does not involve any dimensionful parameters

$$\mathbf{Q}_2(\epsilon) = -\frac{1}{2} \int_{S^{1,1}} d\sigma d\theta \epsilon (2D\Phi^\rho S_\nu J_\rho^\nu + D\Phi^\nu D\Phi^\rho L_{\nu\rho} + S_\nu S_\rho P^{\nu\rho}). \quad (3.16)$$

We can combine $D\Phi$ and S into a single object

$$\Lambda = \begin{pmatrix} D\Phi \\ S \end{pmatrix} \quad (3.17)$$

which can be thought of as a section of pullback of $X^*(\Pi(T \oplus T^*))$. The tensors in (3.16) can be combined into a single object³

$$\mathcal{J} = \begin{pmatrix} -J & P \\ L & J^t \end{pmatrix}, \quad (3.18)$$

which is understood now as $\mathcal{J} : T \oplus T^* \rightarrow T \oplus T^*$. With this new notation we can rewrite (3.16) as follows

$$\mathbf{Q}_2(\epsilon) = -\frac{1}{2} \int_{S^{1,1}} d\sigma d\theta \epsilon \langle \Lambda, \mathcal{J} \Lambda \rangle. \quad (3.19)$$

If the generators $\mathbf{Q}_1(\epsilon)$ and $\mathbf{Q}_2(\epsilon)$ satisfy the algebra (3.13) and (3.15) then we say that there is $N = 2$ supersymmetry. The following proposition tells us when there exists $N = 2$ supersymmetry.

³To relate to other notation (e.g., in [12]), the whole problem is invariant under the change $J \rightarrow -J$.

Proposition 2 $\Pi T^* \mathcal{LM}$ admits $N = 2$ supersymmetry if and only if M is a generalized complex manifold.

Proof: We have to impose the algebra (3.15) on $\mathbf{Q}_2(\epsilon)$. The calculation of the second bracket is lengthy but straightforward. The coordinate expressions coincide with those given in [12]. Therefore we give only the final result of the calculation. Thus the algebra (3.15) satisfied if and only if

$$\mathcal{J}^2 = -1_{2d}, \quad \Pi_{\mp}[\Pi_{\pm}(X + \eta), \Pi_{\pm}(Y + \eta)]_c = 0, \quad (3.20)$$

where $\Pi_{\pm} = \frac{1}{2}(1_{2d} \pm i\mathcal{J})$. Thus (3.20) together with the fact that \mathcal{J} (see (3.18)) respects the natural pairing ($\mathcal{J}^t \mathcal{I} = -\mathcal{I} \mathcal{J}$) implies that \mathcal{J} is a generalized complex structure. Π_{\pm} project to two maximally isotropic involutive subbundles L and \bar{L} such that $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L}$. Thus we have shown that $\Pi T^* \mathcal{LM}$ admits $N = 2$ supersymmetry if and only if M is a generalized complex manifold. Our derivation is algebraic in nature and does not depend on the details of the model.

The canonical transformations of $\Pi T^* \mathcal{LM}$ cannot change any brackets. Thus the canonical transformation corresponding to a b-transform (2.9)

$$\begin{pmatrix} D\Phi \\ S \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} D\Phi \\ S \end{pmatrix} \quad (3.21)$$

induces the following transformation of the generalized complex structure

$$\mathcal{J}_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \quad (3.22)$$

and thus gives rise to a new extended supersymmetry generator. Therefore \mathcal{J}_b is again the generalized complex structure. This is a physical explanation of the behavior of generalized complex structure under b -transform.

Using $\delta_i(\epsilon)\bullet = \{\mathbf{Q}_i(\epsilon), \bullet\}$ we can write down the explicit form for the second supersymmetry transformations as follows

$$\delta_2(\epsilon)\Phi^\mu = \epsilon D\Phi^\nu J_\nu^\mu - \epsilon S_\nu P^{\mu\nu} \quad (3.23)$$

$$\delta_2(\epsilon)S_\mu = \epsilon D(S_\nu J_\mu^\nu) - \frac{1}{2}\epsilon S_\nu S_\rho P^{\nu\rho}_{,\mu} + \epsilon D(D\Phi^\nu L_{\mu\nu}) + \epsilon S_\nu D\Phi^\rho J_{\rho,\mu}^\nu - \frac{1}{2}\epsilon D\Phi^\nu D\Phi^\rho L_{\nu\rho,\mu}. \quad (3.24)$$

Indeed it coincides with the supersymmetry transformation for the topological model analyzed in [12]⁴.

⁴Namely, in [12] the transformations (4.2)-(4.3) subject to (4.5)-(4.6) coincide with (3.23)-(3.24) in the present paper, modulo obvious identifications.

Alternatively we can relate the generalized complex structure with an odd differential δ on $C^\infty(\Pi T^* \mathcal{L}M)$. Indeed the supersymmetry transformations (A.3) and (3.23)-(3.24) can be thought of as odd transformations (by putting formally $\epsilon = 1$) which squares to the translations, ∂ . Thus we can define the odd generator

$$\mathbf{q} = \mathbf{Q}_1(1) + i\mathbf{Q}_2(1) = - \int_{S^{1,1}} d\sigma d\theta (S_\mu Q \Phi^\mu + i D \Phi^\rho S_\nu J_\rho^\nu + \frac{i}{2} D \Phi^\nu D \Phi^\rho L_{\nu\rho} + \frac{i}{2} S_\nu S_\rho P^{\nu\rho}), \quad (3.25)$$

which gives rise to the following transformations

$$\delta \Phi^\mu = Q \Phi^\mu + i D \Phi^\nu J_\nu^\mu - i S_\nu P^{\mu\nu}, \quad (3.26)$$

$$\delta S_\mu = Q S_\mu + i D (S_\nu J_\mu^\nu) - \frac{i}{2} S_\nu S_\rho P^{\nu\rho}_{,\mu} + i D (D \Phi^\nu L_{\mu\nu}) + i S_\nu D \Phi^\rho J_{\rho,\mu}^\nu - \frac{i}{2} D \Phi^\nu D \Phi^\rho L_{\nu\rho,\mu}. \quad (3.27)$$

Thus $\delta^2 = 0$ if and only if \mathcal{J} defined in (3.18) is a generalized complex structure. In doing the calculations one should remember that now δ is odd operation and whenever it passes through an odd object (e.g., D , Q and S) there is extra minus. The existence of odd nilpotent operation (3.26)-(3.27) is related to the topological twist of $N = 2$ algebra (for the related discussion see [8] and [9]). The odd generator (3.25) is reminiscent of the solution of master equations proposed in [14] (see also [15]). However there are principal differences related to the setup and to the definitions of basic operations (e.g., D).

It is straightforward to generalize all results to the case when $H \neq 0$. In all formulas we can generate H by non-canonical transformations

$$\Phi^\mu \rightarrow \Phi^\mu, \quad S_\mu \rightarrow S_\mu + B_{\mu\nu} D \Phi^\nu \quad (3.28)$$

with $H_{\mu\nu\rho} = B_{\mu\nu,\rho} + B_{\nu\rho,\mu} + B_{\rho\mu,\nu}$ with $B_{\mu\nu,\rho} \equiv \partial_\rho B_{\mu\nu}$. The transformation (3.28) is just a technical trick and all final formulas contain only H . Thus the generator of manifest supersymmetry is

$$\mathbf{Q}_1(\epsilon) = - \int_{S^{1,1}} d\sigma d\theta \epsilon (S_\mu + B_{\mu\nu} D \Phi^\nu) Q \Phi^\mu = \int_{S^1} d\sigma \epsilon (p_\mu \lambda^\mu - \rho_\mu \partial X^\mu - \frac{1}{3} H_{\mu\nu\rho} \lambda^\mu \lambda^\nu \lambda^\rho) \quad (3.29)$$

and the generator of translations is

$$\mathbf{P}(a) = \int_{S^{1,1}} d\sigma d\theta a (S_\mu \partial \Phi^\mu - \frac{1}{6} H_{\mu\nu\rho} D \Phi^\mu D \Phi^\nu D \Phi^\rho). \quad (3.30)$$

Assuming the full symplectic structure (2.8), $\mathbf{Q}_1(\epsilon)$ and $\mathbf{P}(a)$ obey the same algebra (3.13) as before. The ansatz for the generator $\mathbf{Q}_2(\epsilon)$ of the second supersymmetry is the same as before (3.16) and the algebra (3.15) should be imposed. In its turn the algebra implies the following conditions

$$\mathcal{J}^2 = -1_{2d}, \quad \Pi_\mp [\Pi_\pm (X + \eta), \Pi_\pm (Y + \eta)]_H = 0, \quad (3.31)$$

where $[\cdot]$ is twisted Courant bracket. Therefore now \mathcal{J} is a twisted generalized complex structure.

There is a possibility to modify the supersymmetry algebra slightly [6]. Namely for the "pseudo-supersymmetry" algebra the last condition in (3.15) is replaced by the following one

$$\{\mathbf{Q}_2(\epsilon), \mathbf{Q}_2(\tilde{\epsilon})\} = -\mathbf{P}(2\epsilon\tilde{\epsilon}). \quad (3.32)$$

Geometrically it implies that

$$\mathcal{J}^2 = 1_{2d}, \quad \Pi_{\mp}[\Pi_{\pm}(X + \eta), \Pi_{\pm}(Y + \eta)]_H = 0, \quad (3.33)$$

where $\Pi_{\pm} = \frac{1}{2}(1_{2d} \pm \mathcal{J})$. Using the fact \mathcal{J} respects the natural pairing we conclude that Π_{\pm} project to maximally isotropic subbundles which are involutive with respect to (twisted) Courant bracket. Therefore we get two complementary (twisted) Dirac structures L_+ and L_- such that $T \oplus T^* = L_+ \oplus L_-$. For any M there is always a trivial "pseudo-supersymmetry"

$$\delta_2(\epsilon)\Phi^{\mu} = \epsilon D\Phi^{\mu}, \quad \delta_2(\epsilon)S_{\mu} = -\epsilon DS_{\mu}, \quad (3.34)$$

which corresponds to the choice $\mathcal{J} = 1_{2d}$. Another interesting example of "pseudo-supersymmetry" is given by the following choice

$$\mathcal{J} = \begin{pmatrix} 1_d & P \\ 0 & -1_d \end{pmatrix}, \quad (3.35)$$

where P is a Poisson structure on M . In this case the transformations are

$$\delta_2(\epsilon)\Phi^{\mu} = \epsilon D\Phi^{\mu} - \epsilon S_{\nu} P^{\mu\nu}, \quad \delta_2(\epsilon)S_{\mu} = -\epsilon DS_{\mu} - \frac{1}{2}\epsilon S_{\nu} S_{\rho} P^{\nu\rho}_{,\mu}. \quad (3.36)$$

In analogy with the discussion of the standard $N = 2$ supersymmetry we can consider the topological twist of "pseudo-supersymmetry". Namely we can introduce an odd nilpotent operation on $\Pi T^* \mathcal{L}M$ as follows $\delta = \delta_1(1) + \delta_2(1)$. Thus for the example (3.35) the corresponding nilpotent operation δ is reminiscent of the BV-transformations of the Poisson sigma model [2] (however we are working in Hamiltonian setup).

The above discussion about the odd transformations can be summarized as follows

Proposition 3 *The super-Poisson algebra $C^{\infty}(\Pi T^* \mathcal{L}M)$ admits an odd differential δ if either M is generalized complex manifold or M is Dirac manifold with two complementary Dirac structures.*

4 D-branes

We can generalize the previous discussion to the world-sheets with the boundary. For the world-sheet $\Sigma = P^1 \times \mathbb{R}$ with P^1 being the interval $[0, 1]$ the phase space can be identified with the cotangent bundle T^*PM of the path space $PM = \{X : [0, 1] \rightarrow M, X(0) \in D_0, X(1) \in D_1\}$ where D_0 and D_1 are submanifolds of M . To write down the symplectic structure on T^*PM we have to require that D_0 and D_1 are generalized submanifolds of M , i.e. there exists $B^i \in \Omega^2(D^i)$ such that $dB^i = H|_{D^i}$. Hence the symplectic structure is given by

$$\begin{aligned} \omega = & \int_0^1 d\sigma \left(\delta X^\mu \wedge \delta p_\mu + \frac{1}{2} H_{\mu\nu\rho} \partial X^\mu \delta X^\nu \wedge \delta X^\rho \right) + \\ & + \frac{1}{2} B_{\mu\nu}^0(X(0)) \delta X^\mu(0) \wedge \delta X^\nu(0) - \frac{1}{2} B_{\mu\nu}^1(X(1)) \delta X^\mu(1) \wedge \delta X^\nu(1), \end{aligned} \quad (4.37)$$

where the boundary contributions are needed in order ω to be closed.

Next we proceed with the supersymmetrization of T^*PM . In analogy with the previous discussion we introduce the "superinterval" $P^{1,1}$ with coordinates (σ, θ) and the superinterval space $\mathcal{P}M = \{\Phi : P^{1,1} \rightarrow M\}$. The phase space is given by the cotangent bundle $\Pi T^*\mathcal{P}M$ of $\mathcal{P}M$, with reversed parity on the fibers. As before we introduce two superfields Φ and S , see (2.7). Let us start to discuss the situation when $H = 0$. The canonical symplectic structure on $\Pi T^*\mathcal{P}M$ is given by

$$\omega = \int_{P^{1,1}} d\sigma d\theta \delta\Phi^\mu \wedge \delta S_\mu. \quad (4.38)$$

However this symplectic form is not supersymmetric unless the extra data is specified. In presence of boundaries the superfield expressions are not automatically supersymmetric due to possible boundary terms. Namely the transformation of symplectic form (4.38) under manifest supersymmetry (A.3) gives rise to a boundary term

$$\delta_1(\epsilon)\omega = \epsilon(\delta X^\mu(1) \wedge \delta\rho_\mu(1) - \delta X^\mu(0) \wedge \delta\rho_\mu(0)). \quad (4.39)$$

Moreover the supersymmetry algebra has boundary term

$$\{\mathbf{Q}_1(\epsilon), \mathbf{Q}_1(\tilde{\epsilon})\} = P(2\epsilon\tilde{\epsilon}) + 2(\rho_\mu(1)\lambda^\mu(1) - \rho_\mu(0)\lambda^\mu(0)) \quad (4.40)$$

and $\mathbf{Q}_1(\epsilon)$ is not translation-invariant

$$\{\mathbf{P}(a), \mathbf{Q}_1(\epsilon)\} = a\epsilon(p_\mu(1)\lambda^\mu(1) - p_\mu(0)\lambda^\mu(0) + \rho_\mu(0)\partial X^\mu(0) - \rho_\mu(1)\partial X^\mu(1)). \quad (4.41)$$

These boundary terms spoil wanted properties. The problem can be cured by imposing the appropriate boundary conditions on the fields such that the boundary terms vanish. In fact the required boundary conditions have a simple geometrical form

$$\Lambda(1) = \begin{pmatrix} D\Phi(1) \\ S(1) \end{pmatrix} \in X^*(\Pi(TD_1 \oplus N^*D_1)), \quad \Lambda(0) = \begin{pmatrix} D\Phi(0) \\ S(0) \end{pmatrix} \in X^*(\Pi(TD_0 \oplus N^*D_0)) \quad (4.42)$$

where TD_i is tangent and N^*D_i is conormal bundles of D_i correspondingly. The conditions (4.42) are understood as conditions on each component of superfields.

Next we consider the case when $H \neq 0$. We should apply the same logic as before. Namely we have to impose such boundary conditions on the fields that there are no unwanted boundary terms which spoil supersymmetry. Consequently we arrive to the following symplectic form for $\Pi T^*\mathcal{P}M$

$$\begin{aligned} \omega = & \int_{P^{1,1}} d\sigma d\theta (\delta\Phi^\mu \wedge \delta S_\mu - \frac{1}{2} H_{\mu\nu\rho} D\Phi^\mu \delta\Phi^\nu \wedge \delta\Phi^\rho) + \\ & + \frac{1}{2} B_{\mu\nu}^0(X(0)) \delta X^\mu(0) \wedge \delta X^\nu(0) - \frac{1}{2} B_{\mu\nu}^1(X(1)) \delta X^\mu(1) \wedge \delta X^\nu(1), \end{aligned} \quad (4.43)$$

with the fields satisfying the following boundary conditions

$$\Lambda(1) \in X^*(\Pi\tau_{D_1}^{B^1}), \quad \Lambda(0) \in X^*(\Pi\tau_{D_0}^{B^0}), \quad (4.44)$$

where $\tau_{D_i}^{B^i}$ is a generalized tangent bundle (B.8) of generalized submanifold (D_i, B^i) . With these boundary conditions the symplectic form (4.43) is supersymmetric and there are no boundary terms in the supersymmetry algebra. Actually the boundary conditions (4.44) can be thought of as B -transform of the conditions (4.42). The spaces $\tau_{D_i}^{B^i}$ are maximally isotropic with respect to the natural pairing $\langle \cdot, \cdot \rangle$ on $T \oplus T^*$, i.e.

$$\langle \Lambda(0), \Lambda(0) \rangle = 0, \quad \langle \Lambda(1), \Lambda(1) \rangle = 0. \quad (4.45)$$

Finally we have constructed the the supersymmetric version of $\Pi T^*\mathcal{P}M$ where the boundary conditions (4.44) play the crucial role.

Now we turn to the discussion of extended supersymmetry. As in previous section we should write down the generator for second supersymmetry (3.16) and check the algebra (3.15). The only difference with the discussion from section 3 is that we should keep track of the boundary terms. For example, we can check if $\mathbf{Q}_2(\epsilon)$ is translation-invariant, i.e.

$$\{\mathbf{P}(a), \mathbf{Q}_2(\epsilon)\} = \frac{1}{2}a \int d\theta \epsilon \langle \Lambda, \mathcal{J}\Lambda \rangle(0) - \frac{1}{2}a \int d\theta \epsilon \langle \Lambda, \mathcal{J}\Lambda \rangle(1). \quad (4.46)$$

Thus translation-invariance is spoiled by the boundary terms. We can restore it by imposing the additional property

$$\langle \Lambda, \mathcal{J}\Lambda \rangle(0) = 0, \quad \langle \Lambda, \mathcal{J}\Lambda \rangle(1) = 0. \quad (4.47)$$

Indeed this property (together with (4.44)) is sufficient to cancel all other unwanted boundary terms, e.g. in (3.15) or in $\delta_2(\epsilon)\omega$. Thus the property (4.47) together with (4.44) implies that the subbundles $\tau_{D_i}^{B_i}$ are stable under \mathcal{J} , i.e. (D_i, B_i) are the generalized complex manifolds introduced in [4].

We summarize the above discussion in proposition

Proposition 4 *The cotangent bundle $\Pi T^*\mathcal{P}M$ admits $N = 2$ supersymmetry if and only if M is (twisted) generalized complex manifold and (D_i, B_i) are generalized complex submanifolds.*

This result agrees with the previous considerations in [10] and [13]. However now it is applicable to a wide class of sigma models.

If instead we consider the “pseudo-supersymmetry” algebra (3.32) then appropriate boundary conditions would require that $\tau_{D_i}^{B_i}$ is invariant under \mathcal{J} which defines two transversal Dirac structures. This is a real analog of the generalized complex submanifold. In the example (3.35), τ_D^0 is stable under \mathcal{J} if a submanifold D is coisotropic with respect to P , see [3].

5 Concluding remarks

In this short note we clarified and extended results from [12]. The first order actions discussed in [11] and [12] can be thought of as phase space actions and therefore the Hamiltonian formalism should naturally arise in the problem. Indeed the Hamiltonian formalism offers a deep insight on the relation between the world-sheet and the geometry of $T \oplus T^*$. The main result of the paper is that the phase space realization of extended supersymmetry is related to generalized complex structure. This result is model independent and it is applicable to the wide range of sigma models, e.g. the standard sigma model, the Poisson sigma model, the twisted Poisson sigma model etc.

The next step would be to specify the Hamiltonian \mathcal{H} (i.e., choose the concrete model) and check that \mathbf{Q}_2 is in fact the symmetry of the Hamiltonian. At this stage the compatibility between the geometrical data used in \mathcal{H} (e.g., a metric g , a Poisson structure π etc.) and \mathcal{J} will arise. We hope to come back to this elsewhere.

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A Appendix: superspace conventions

We use the superspace conventions. The odd coordinate is labeled by θ and the covariant derivative D and supersymmetry generator Q are defined as follows

$$D = \partial_\theta - \theta \partial_\sigma, \quad Q = \partial_\theta + \theta \partial_\sigma \quad (\text{A.1})$$

such that

$$D^2 = -\partial_\sigma, \quad Q^2 = \partial_\sigma, \quad QD + DQ = 0. \quad (\text{A.2})$$

In terms of covariant derivatives, a supersymmetry transformation⁵ of a superfields is then given by

$$\delta_1(\epsilon)\Phi^\mu \equiv \epsilon Q\Phi^\mu, \quad \delta_1(\epsilon)S_\mu \equiv \epsilon QS_\mu. \quad (\text{A.3})$$

The components of superfields can be found via projection as follows,

$$\Phi| \equiv X^\mu, \quad D\Phi| \equiv \lambda^\mu, \quad S_\mu| \equiv \rho_\mu, \quad DS_\mu| \equiv p_\mu, \quad (\text{A.4})$$

where a vertical bar denotes “the $\theta = 0$ part of”. Thus, in components, the N=1 supersymmetry transformations are given by

$$\delta_1(\epsilon)X^\mu = \epsilon\lambda^\mu, \quad \delta_1(\epsilon)\lambda^\mu = -\epsilon\partial X^\mu, \quad \delta_1(\epsilon)\rho_\mu = \epsilon p_\mu, \quad \delta_1(\epsilon)p_\mu = -\epsilon\partial\rho_\mu. \quad (\text{A.5})$$

The N=1 spinorial measure in terms of covariant derivatives

$$\int d\theta \mathcal{L} = D\mathcal{L}| \quad (\text{A.6})$$

B Appendix: basics on $T \oplus T^*$

Consider the vector bundle $T \oplus T^*$ which is the sum of the tangent and cotangent bundles of an d -dimensional manifold M . $T \oplus T^*$ has a natural pairing

$$\langle X + \xi, Y + \eta \rangle \equiv (i_Y \xi + i_X \eta) \equiv \begin{pmatrix} X \\ \xi \end{pmatrix}^t \mathcal{I} \begin{pmatrix} Y \\ \eta \end{pmatrix}. \quad (\text{B.1})$$

⁵We give the expressions for the case $H = 0$. Analogously using the generator (3.29) one can write down the expressions for the case $H \neq 0$.

The smooth sections of $T \oplus T^*$ have a natural bracket operation called the Courant bracket and defined as follows

$$[X + \xi, Y + \eta]_c = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d(i_X \eta - i_Y \xi) \quad (\text{B.2})$$

where $[\cdot, \cdot]$ is a Lie bracket on T . Given a closed three form H we can define a twisted Courant bracket

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta]_c + i_X i_Y H. \quad (\text{B.3})$$

The orthogonal automorphism (i.e., such which preserves $\langle \cdot, \cdot \rangle$) $F : T \oplus T^* \rightarrow T \oplus T^*$ of (twisted) Courant bracket

$$F([X + \xi, Y + \eta]_H) = [F(X + \xi), F(Y + \eta)]_H \quad (\text{B.4})$$

is semidirect product of $Diff(M)$ and $\Omega_{closed}^2(M)$, where the action of the closed two form is given as follows

$$e^b(X + \xi) \equiv X + \xi + i_X b \quad (\text{B.5})$$

for $b \in \Omega_{closed}^2(M)$. The transformation (B.5) is called b -transform. The maximally isotropic subbundle L of $T \oplus T^*$, which is involutive with respect to (twisted) Courant bracket is called (twisted) Dirac structure. We can consider two complementary (twisted) Dirac structures L_+ and L_- such that $T \oplus T^* = L_+ \oplus L_-$. Alternatively we can define L_{\pm} by proving a map $\mathcal{J} : T \oplus T^* \rightarrow T \oplus T^*$ with the following properties

$$\mathcal{J}^t \mathcal{I} = -\mathcal{I} \mathcal{J}, \quad \mathcal{J}^2 = 1_{2d}, \quad \Pi_{\mp}[\Pi_{\pm}(X + \xi), \Pi_{\pm}(Y + \eta)]_H = 0 \quad (\text{B.6})$$

where $\Pi_{\pm} = \frac{1}{2}(1_{2d} \pm \mathcal{J})$ are projectors on L_{\pm} .

The (twisted) generalized complex structure is the complex version of two complementary (twisted) Dirac subbundles such that $(T \oplus T^*) \oplus \mathbb{C} = L \oplus \bar{L}$. We can define the generalized complex structure as a map $\mathcal{J} : (T \oplus T^*) \otimes \mathbb{C} \rightarrow (T \oplus T^*) \otimes \mathbb{C}$ with the following properties

$$\mathcal{J}^t \mathcal{I} = -\mathcal{I} \mathcal{J}, \quad \mathcal{J}^2 = -1_{2d}, \quad \Pi_{\mp}[\Pi_{\pm}(X + \xi), \Pi_{\pm}(Y + \eta)]_H = 0, \quad (\text{B.7})$$

where $\Pi_{\pm} = \frac{1}{2}(1_{2d} \pm i\mathcal{J})$ are the projectors on L and \bar{L} correspondingly.

The generalized submanifold is a submanifold D with a two form $B \in \Omega^2(M)$ such that $dB = H|_D$. For generalized submanifold (D, B) we can define the generalized tangent bundle

$$\tau_D^B = \{X + \xi \in TD \oplus T^*M|_D, \quad \xi|_D = i_X B\}. \quad (\text{B.8})$$

The submanifold (D, B) is (twisted) generalized complex submanifold if τ_D^B is stable under the action of map \mathcal{J} defined in (B.7).

For further details the reader may consult the Gualtieri's thesis [4].

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